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FINAL REPORT

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"Fractal and Multifractal Approaches to Clustering Phenomena"

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Fractal and Multifractal Approaches to Clustering Phenomena

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Under the auspices of this contract, we have attempted to achieve some genuine understanding of diffusion limited aggregation (DLA), the paradigm model for dynamical mechanisms of disorderly growth processes [1-10]. We found that while the growth probabilities for the tips of the DLA structure do scale in the conventional fashion, there is evidence that the growth probabilities of the fjords do *not* scale. Does this competition between one part of DLA that does scale, and another that does not, underlie many of the unusual properties of this model?

1. Introduction

We will organize the presentation around three questions:
Question 1: "*What is DLA?*"
Question 2: "*Why are we interested?*"
Question 3: "*What did we actually do?*"

2. First Question: "What is Diffusion-Limited Aggregation?"

Like many models in statistical mechanics, the rule defining DLA is simple [11]. At time 1, we place in the center of a computer screen a white pixel, and release a random walk from a large circle surrounding the white pixel (Fig. 1a). The four perimeter sites have an equal *a priori* probability p_i to be stepped on by the random walk; accordingly we write

$$p_i = \frac{1}{4} \qquad (i = 1, \ldots, 4). \tag{1}$$

The rule is that the random walker sticks irreversibly—thereby forming a cluster of mass $M = 2$. There are $N_p = 6$ possible sites, henceforth called *growth* *sites* (Fig. 1b), but now the probabilities are *not* all identical: each of the growth sites of the two tips has growth probability $p_{\max} \cong 0.22$, while each of the four growth sites on the sides has growth probability $p_{\min} \cong 0.14$. Since a side on the tip is 50% more likely to grow than a site on the sides, the next site is more likely to be added to the tip—it is like capitalism in that "the rich get richer." One of the main features of our approach to DLA is that instead of focusing on the tips who are "getting richer", we can focus on the fjords who are "getting poorer"—which is perhaps more familiar to physicists who all know the feeling that "once you get behind you stay behind!"

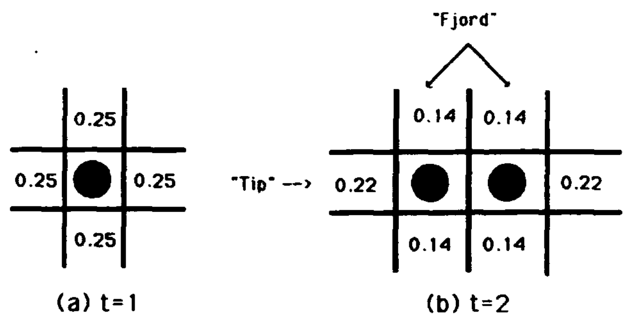

Fig. 1. (a) Square lattice DLA at time $t = 1$, showing the four growth sites, each with growth probability $p_i = 1/4$. (b) DLA at time $t = 2$, with 6 growth sites, and their corresponding growth probabilities p_i.

Just because the third particle is *more likely* to stick at the tip does not mean that the next particle *will* stick on the tip. Indeed, the most that one can say about the cluster is to specify the *growth site probability distribution* (GSPD)—i.e., the set of numbers,

$$\{p_i\} \qquad i = 1 \ldots N_p, \tag{2}$$

where p_i is the probability that perimeter site ("growth site") i is the next to grow, and N_p is the total number of perimeter sites ($N_p = 4, 6$ for the cases $M = 1, 2$ shown in Figs. 1a and 1b respectively). The recognition that the set of $\{p_i\}$ gives us essentially the *maximum* amount of information we can have about the system is connected to the fact that tremendous attention has been paid to these p_i—and to the analogs of the p_i in various closely-related systems [12-20].

If the DLA growth rule is simply iterated , then we obtain a large cluster characterized by a range of growth probabilities that spans several orders of magnitude—from the tips to the fjords. Figure 2 shows such a large cluster, where each pixel is colored according to the time it was added to the aggregate. From the fact that the "last to arrive" particles (green pixels) are never found to be adjacent to the "first to arrive" particles (white pixels), we conclude that the p_i for the growth sites on the tips must be vastly larger than the p_i for the growth sites in the fjords.

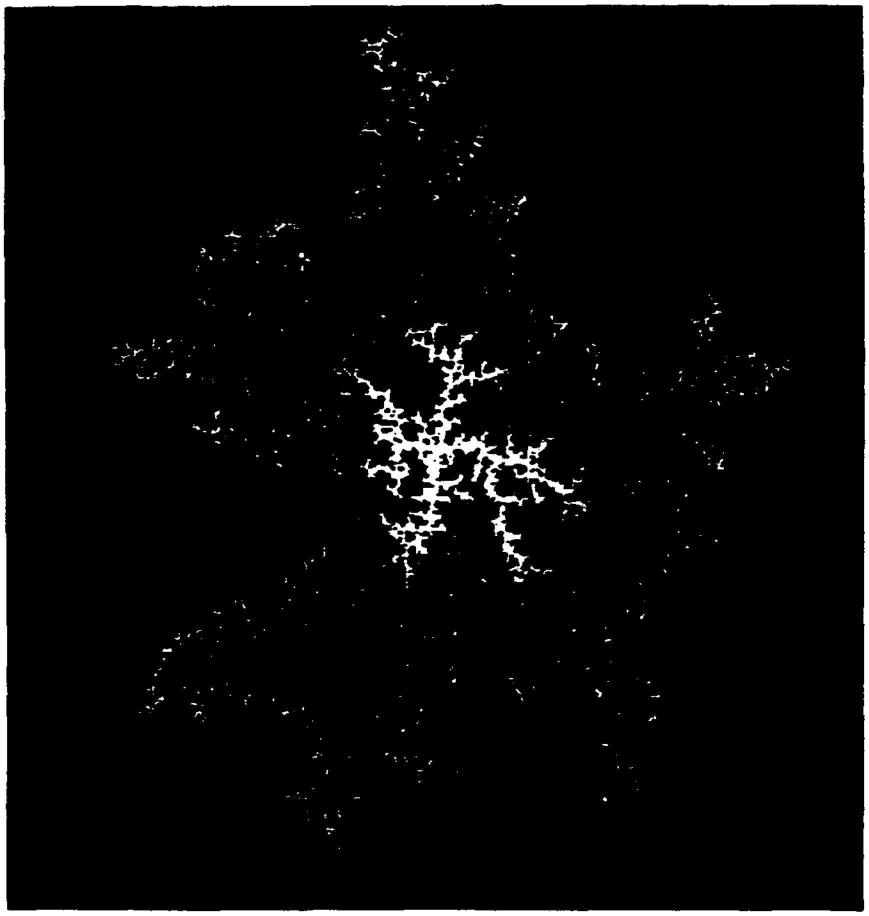

Fig. 2. Large DLA cluster on a square lattice. Each cluster site is color-coded according to the time which the site joined the cluster. Courtesy of P. Meakin.

3. Second Question: "Why Study DLA?"

There are almost always two reasons why one finds a given model interesting, and hence there are generically two distinct answers to this question.

A. Answer One: "There Are Experimental Realizations"

Today, there are roughly of order 10^2 systems in nature for which DLA may be relevant [21-24]. Indeed, it seems that possibly DLA captures the essential physics of a *typical* dynamic growth process that can be related to the Laplace Equation (with appropriate boundary conditions).

First is the fact that aggregation phenomena based on random walkers leads to a Laplace equation for the probability $\Pi(r,t)$ that a walker is at position r and time t [25]. More surprising, however, is the vast range of phenomena [21-24] that at first sight seem to have nothing to do with random walkers. These include fluid-fluid displacement phenomena ("viscous fingers"), for which the pressure P at every point satisfies a Laplace equation $\nabla^2 P = 0$

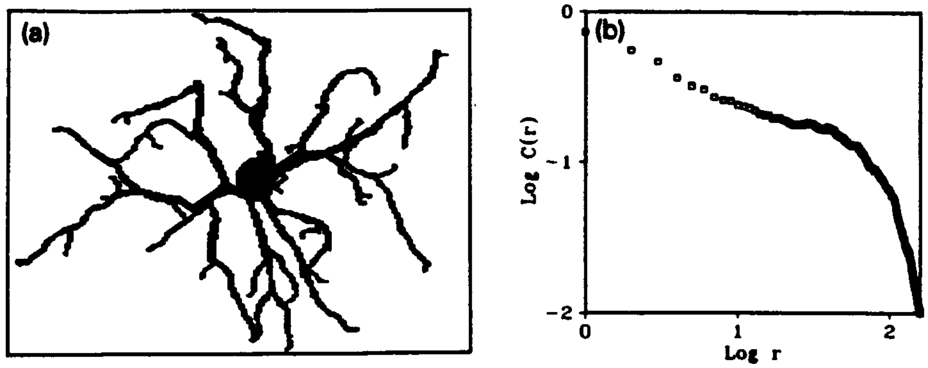

Fig. 3. Typical retinal neuron and its fractal analysis. From Ref. [36].

[26-28]. Similarly, dielectric breakdown phenomena [29], chemical dissolution [30], electrodeposition [31], and a host of other displacement phenomena (including even dendritic crystal growth [32] and snowflake growth [33]) may be members of a suitably-defined *DLA universality class*.

Recently, several phenomena of *biological* interest have attracted the attention of DLA *afficionados*. These include the growth of bacterial colonies [34], the retinal vasculature [35], and neuronal outgrowth [36]. The last example is particularly intriguing since if evolution chose DLA as the morphology for the nerve cell (Fig. 3), then perhaps we can understand "why" this choice was made. What evolutionary advantage does a DLA morphology convey? Can we use the answer to this question to better design the next generation of computers? These are important issues that we hope to address between this and the next Bar-Ilan conference, but already we appreciate that a fractal object is the most efficient way to obtain a great deal of intercell "connectivity" with a minimum of "cell volume", so the next question is "which" fractal did evolution select, and why?

We will save time and space by resisting the temptation at this point to "pull out the family photo album" to show lots of all these various realizations. Instead, we may refer the interested reader (and their non-specialist colleagues) to the forthcoming *Album of Fractal Forms* [37].

B. Answer Two: "Understanding DLA Growth is a Theoretical Challenge"

As with many models in statistical mechanics, the theoretical challenge is as important as the experimental realizations in "hooking" theorists. And as with many statistical mechanical models, the "defining rule" in DLA is simple even though the "consequences of that rule" are extremely rich. Understanding how such a rich consequence can follow from such a simple rule is indeed an irresistible challenge.

In the case of DLA, this challenge is enhanced by the fact that—unlike other models with simple rules (such as the Ising model)— in DLA there is no Boltzmann factor so we can more easily explain and understand since one does not have to know any physics beforehand. Indeed, it initially surprises almost everyone who sees DLA develop in real time on a computer screen that a complex outcome (at the *global* level of a "form") seems to bear no obvious relation to the details of the simple *local* rule that produced this form.

There is even the philosophical challenge of understanding how it is that even though no two DLA's are identical (in the same sense that we can say no two snowflakes are identical), nonetheless every DLA that we are likely to ever see has a generic "form" that even a child can recognize (in the same sense that almost every snowflake that we are likely to see has a generic form that every child recognizes).

A second somewhat "philosophical" point is the following. If we understand the essential physics of an extremely robust model, such as the Ising model, then we say that we understand the essential physics of the complex materials that fall into the universality class described by the Ising model. In fact, by understanding the pure Ising model, we can even understand most of the features of *variants* of the Ising model (such as the XY or Heisenberg models) that may be appropriate for describing even more complex materials. Similarly, we feel that if we can understand DLA, then we are well on our way to understanding *variants* of DLA—such as DLA grown on a lattice ("DLA with anisotropy") or DLA grown using the noise reduction algoritm [38]. And just as the Ising model is a paradigm for all systems composed of interacting subunits, so also DLA may be a paradigm for all kinetic growth models.

So with these ambitious goals, we now proceed to consider the third question.

4. Third Question: "What Do We Actually Do?"

A. Fractal Dimension: Straightforward to Calculate, but Fruitless

Until relatively recently, most of the theoretical attention paid to DLA has focussed on its fractal properties [39-40]. One definition of the fractal dimension d_f is by the "window box scaling" operation:

(1) First place an imaginary window box of edge L around an arbitrarily-chosen occupied DLA site ("local origin").

(2) Then count the number of occupied pixels $M(L)$ within that window box.

(3) Next choose many different local origins to obtain good statistics.

(4) Finally, make a log-log plot of $M(L)$ *vs.* L, and interpret the fractal dimension d_f as the "asymptotic" ($L \to \infty$) slope of this plot.

Conventionally, we write

$$M(L) \sim L^{d_f}, \tag{3}$$

where the tilde denotes "asymptotically equal to."

The difficulty of extrapolating from finite L to infinite L has motivated ever more clever algorithms for generating ever larger DLA clusters. Most of the world records are held by P. Meakin and his collaborators [39]:

$$M_{\max} = \begin{cases} 12 \times 10^6 & \text{[square lattice DLA]} \\ 10^6 & \text{[off-lattice DLA]} \end{cases} . \tag{4}$$

The corresponding estimates for d_f are roughly [39]

$$d_f = \begin{cases} 1.55 & \text{[square lattice DLA]} \\ 1.715 \pm 0.004 & \text{[off-lattice DLA]} \end{cases} \tag{5}$$

The result for square lattice DLA is based on theoretical arguments [39], for the simulations themselves are not conclusive in that the estimate of d_f simply decreases slowly with increasing cluster mass. Equation (5) suggests that "anisotropic DLA" (DLA grown on a lattice) is in a different universality class; the calculations discussed below are for small mass ($M < 2600$) for which the influence of the lattice anisotropy is (hopefully!) negligible. We can actually "see with our eyes" that $d_f \simeq 1.7$ by means of a simple *hands-on* demonstration. We begin with a large DLA cluster (Fig. 2). Suppose we take a sequence of boxes with $L = 1, 10, 100$ (in units of the pixel size), and estimate the fraction of the box that is occupied by the DLA. This fraction is called the density,

$$\rho(L) \equiv M(L)/L^d, \tag{6}$$

where $d = 2$ here. Combining (3) and (6), we find

$$\rho(L) \sim L^{d_f - d}. \tag{7}$$

Now (7) is equivalent to the functional equation [40]

$$\rho(\lambda L) = \lambda^{d_f - d} \rho(L). \tag{8}$$

Carrying out this operation on Fig. 2 with $\lambda = 10$ will reveal (Fig. 4)

$$\rho(L) \cong \begin{cases} 1 & L = 1 \\ 1/2 & L = 10 \\ 1/4 & L = 100 \end{cases} . \tag{9}$$

Here the result of (9),

$$\rho(10L) \cong \frac{1}{2}\rho(L),$$

convinces one that $10^{d_f - 2} \cong 1/2$. So

$$d_f - 2 \cong \log_{10} \frac{1}{2} = -0.301, \tag{10a}$$

leading to

$$d_f \cong 1.70. \tag{10b}$$

Although we now have estimates of d_f that are accurate to roughly 1%, we lack any way to *interpret* this estimate. This is in contrast to both the

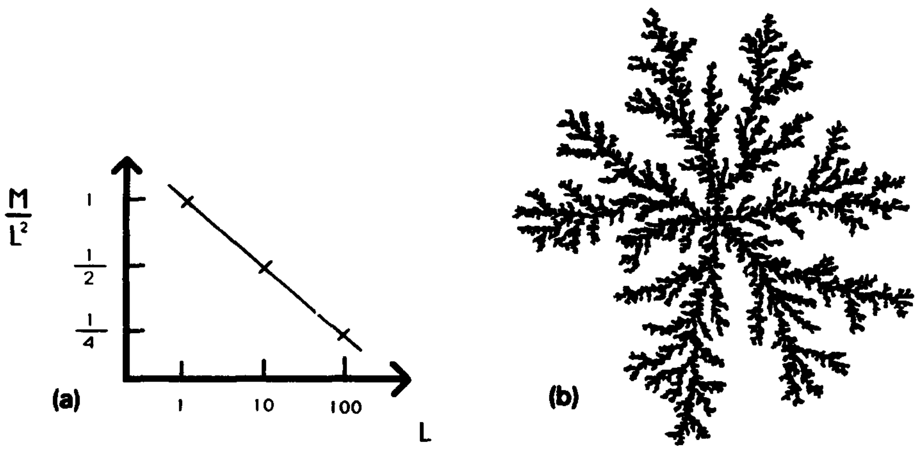

Fig. 4. Schematic illustration of the results of a hands-on experiment to actually see (a) that DLA is indeed a fractal since the density decreases linearly with the size L of the observation window (or inverse wave vector q^{-1}), and (b) that the fractal dimension is given by roughly $d_f - d \simeq \log_{10} \frac{1}{2} \simeq -0.301$.

$d = 2$ Ising model and $d = 2$ percolation, where we can calculate the various exponents and interpret them in terms of scaling powers [40]. Thus [40, 41]

$$y_h = \frac{15}{8} \qquad y_T = 1 \qquad \text{[Ising model]}, \tag{11a}$$

and [41]

$$y_h = d_f = \frac{91}{48}, \qquad y_T = d_{\text{red}} = \frac{3}{4} \qquad \text{[percolation]}, \tag{11b}$$

where d_{red} is the fractal dimension of the singly-connected "red" bonds of the incipient infinite cluster[41].

B. Multifractal Approaches: Complex but Fruitful

Multifractal approaches in statistical physics have a rich history [24], and were first introduced for describing DLA in 1985 by Meakin and collaborators [18]. The key idea is to focus on the set of growth probabilities $\{p_i\}$ and how their distribution function $\mathcal{D}(p_i)$ changes as the cluster mass M increases. The basic reason why this approach *is* fruitful is that the $\{p_i\}$ contains almost the maximum information we can possibly obtain about the dynamics of the growth of DLA. Indeed, specifying the $\{p_i\}$ is analogous to specifying the four "growth" probabilities $p_i = 1/4$ $[i = 1 \cdots 4]$ for a random walker on a square lattice.

The set of numbers $\{p_i\}$ may be used to construct a histogram $\mathcal{D}(\ln p_i)$ shown schematically in Fig. 5. This distribution function can be described by its *moments*, or simply by its *minimum* and *maximum*.

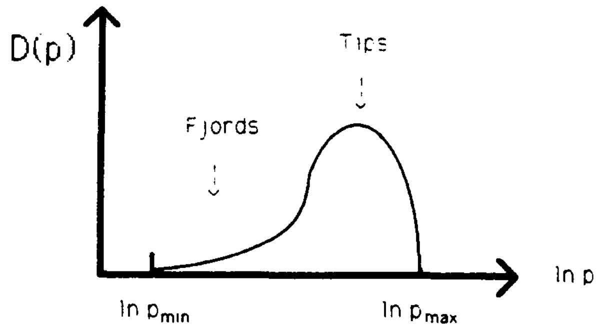

Fig. 5. Schematic behavior of the histogram giving the number $\mathcal{D}(\ln p_i)$ of growth sites with $\ln p_i$ in the interval $[\ln p_i, \ln p_i + \delta \ln p_i]$.

C. Moments of the Distribution Function

The moments of $\mathcal{D}(\ln p_i)$ are determined by

$$Z_\beta = \sum_{\ln p} \mathcal{D}(\ln p) e^{-\beta(-\ln p)}, \tag{12a}$$

or, equivalently,

$$Z_\beta = \sum_i p_i^\beta. \tag{12b}$$

The form (12a) as well as the notation used suggests that we think of β as an inverse temperature, $-\ln p/\ln L$ as an energy, and Z_β as a partition function Accordingly, it is customary to define a "free energy" $F(\beta)$ by the relation

$$Z_\beta = L^{-F(\beta)}, \tag{13a}$$

or, equivalently,

$$F(\beta) = -\frac{\log Z_\beta}{\log L}. \tag{13b}$$

In the literature there exist other symbols, and a brief dictionary is presented in Table 1.

Table 1. Comparison of notation of this paper and other notation in use. Adapted from Ref. [2]

$\beta \longleftrightarrow q$	$F(\beta) \longleftrightarrow \tau(q)$
$E \longleftrightarrow \alpha$	$S(E) \longleftrightarrow f(\alpha)$

What to do with this *thermodynamic formalism*? One approach that we have found to be particularly revealing is the analog for DLA of the successive approximation ("series expansion") approach pioneered by Professor Domb and his collaborators. In fact a Boston University graduate student, J. Lee, recently extended renormalization ideas of Nagatani [43] to actually obtain exact results for a $L \times L$ cell for a sequence of values of L up to and including $L = 5$. This work is described elsewhere [2], so we focus on one key result—the apparent singularity in the quantity

$$C(\beta) \equiv -\frac{\partial^2 F(\beta)}{\partial \beta^2}. \tag{14}$$

Figure 6, which shows $C(\beta)$ for a sequence of L values, is reminiscent of the famous *finite-size-scaling* plot of $C_H(\beta)$ for the $L \times L$ Ising model made by one of Professor Domb's former students, Michael Fisher [44]. Lee interpreted the maximum in $C(\beta)$ as heralding the existence of a singularity in $C(\beta)$ at some critical value β_c (Fig. 7).

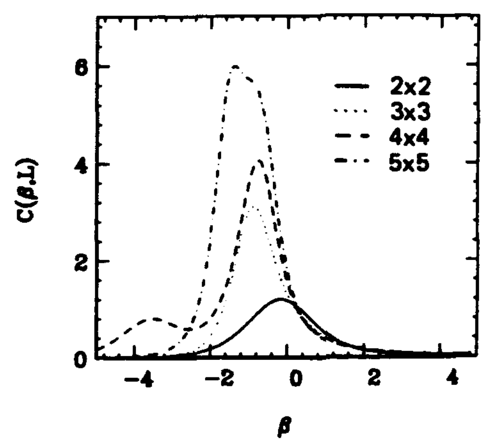

Fig. 6. Analog of Fisher/Ferdinand plot for DLA. Shown is the dependence on β of $\partial^2 F/\partial\beta^2$, where $F \equiv -\log Z/\log L$ and $Z_\beta \equiv \sum_i (p_i)^\beta$. From Ref. [2].

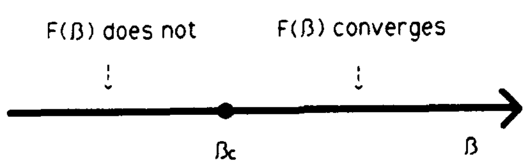

Fig. 7. Schematic illustration of the phase diagram for DLA as a function of the "control parameter" β.

What is the origin of this phase transition, if indeed such a phase transition exists? This question was addressed by Blumenfeld and Aharony (BA) [3]. BA considered the behavior of $p_{\min}$ (the *smallest* of all the growth probabilities $\{p_i\}$) for a typical DLA cluster. BA made the *Ansatz* that

$$p_{\min} \sim e^{-AM^{\alpha}}, \qquad \text{[BA } \textit{Ansatz}\text{]} \tag{15}$$

BA noted that (15) implies that there is a phase transition, with $\beta_c = 0$, since for all *negative* β, the moments Z_β will be dominated by the smallest value of p

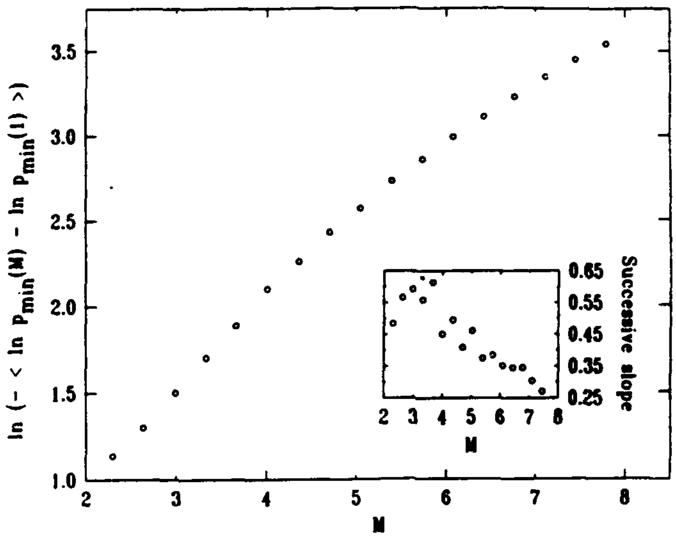

Fig. 8. "Aharony plot" for the dependence of $p_{\min}$ on cluster size (see text). From Ref. [8].

$$Z_\beta \simeq (p_{\min})^\beta \sim e^{AM^x\beta}. \qquad [M \to \infty] \tag{16}$$

Since an exponential is not a power law,

$$e^{-\beta AM^x} \neq M^{-F(\beta)/d_f}, \tag{17}$$

it follows that the free energy of (13) is not defined for $\beta < 0$.

It is not difficult to construct DLA configurations for which the BA *Ansatz* is valid. For example, if a DLA has a tunnel with depth N_T, then the only way a random walker can reach the end of the tunnel is to make a "correct" choice at each site. For a square lattice, a "correct step" will occur with probability 1/4, so

$$p_{\min} = \left(\frac{1}{4}\right)^{N_T} = \exp(-N_T \ln 4). \tag{18}$$

Hence (16) is confirmed, since we expect for those configurations with the longest tunnel lengths that $N_T \sim M^x$ (e.g., for those DLA configurations shaped like a spiral galaxy, we expect $N_T \sim L^2 \sim M^{2/d_f}$).

Recently we decided to search for numerical evidence to test the BA *Ansatz* [8]. To this end, S. Schwarzer and J. Lee calculated the $\{p_i\}$ for approximately 200 DLA clusters of mass about 2600 [8]. This is more than an order of magnitude larger than the size of clusters for which others had evaluated $\{p_i\}$ accurately [19]. The reason for the improvement is that Schwarzer and Lee used an exact enumeration approach [45] whereby one calculates *exactly* the probability that a random walker is at position $\mathbf{r}$ at time t given its probabilities to be at the neighbor sites of $\mathbf{r}$ at time $t-1$.

To test for the form (15), one must plot $p_Q \equiv \exp\langle \ln p_{\min}\rangle$ (which we shall henceforth denote simply $p_{\min}$ against M on log-log paper. We found [Fig. 8] that the data are linear for roughly a decade of mass ($260 \leq M \leq 2600$).

Enter Brooks Harris. Harris [7] noted that such "tunnel configurations" might be sufficiently rare that they do ***not*** make a memorable contribution the quenched average p_Q. Accordingly, Harris proposed that

$$p_{\min}(L) \sim M^{-\alpha}. \qquad \text{[Harris } \textit{Ansatz}\text{]} \tag{19}$$

The Harris Ansatz is supported by a simple deterministic fractal model DLA proposed by Mandelbrot and Vicsek [4]. Equation (16) is replaced by

$$Z_\beta \sim p_{\min}^{\beta} \sim M^{-\alpha\beta}, \tag{20}$$

so there is no phase transition. To test the Harris *Ansatz* (19), we plot p_Q against M on log-log paper and find [Fig. 9] that the data are ***just as linear as for the BA plot***—for roughly the same decade in mass ($260 \leq M \leq 2600$).

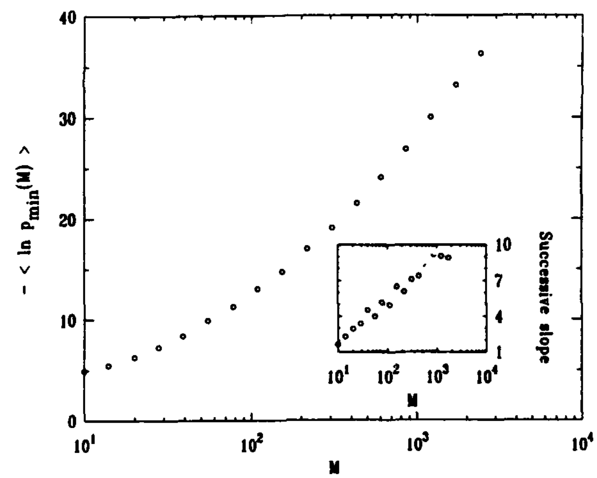

Fig. 9. "Harris plot" for the dependence of $p_{\min}$ on cluster size (see text). From Ref. [8].

So what is going on? Is the decay exponential (as proposed by BA) or is it power law (as proposed by Harris)? To answer this question, we show in Figs. 8, 9 the ***successive slopes*** of both the BA and Harris plots and note that these quantities decrease in the former case and increase in the latter case. This fact strongly suggests that at large mass,

$$\log M \ll \log p_{\min} \ll M^{x}. \tag{21}$$

More significantly, we note that for the Harris plot the slopes increase approximately linearly with $\log M$,

$$\frac{\partial}{\partial(\log M)}(\log p_{\min}) \sim (\log M). \tag{22a}$$

Even a physicist can solve the differential equation (22a),

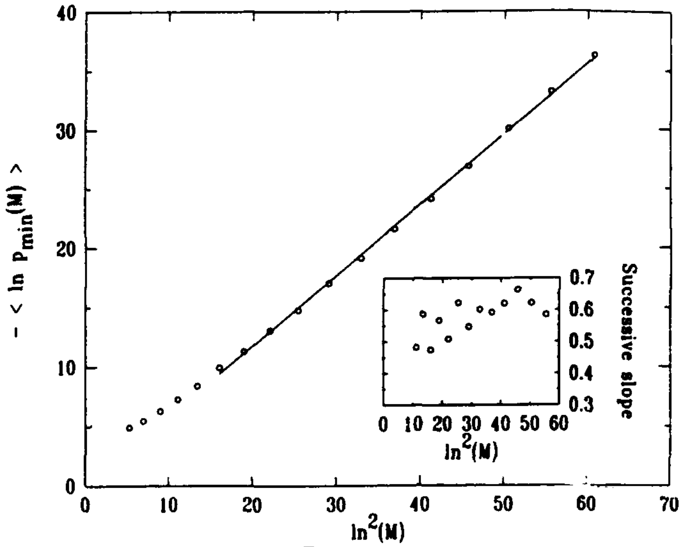

Fig. 10. Plot in the form suggested by linear increase in the successive slopes of the Harris plot of Fig. 9. From Ref. [8].

$$\log p_{\rm min} \sim (\log M)^2, \tag{22b}$$

suggesting that we plot $\log p_{\rm min}$ against $(\log M)^2$. We find linearity over fully two decades, for $26 \leq M \leq 2600$ [Fig. 10], instead of the linearity over only one decade found when testing the BA and Harris assumptions.

D. The Void-Channel Model of DLA Growth

Does the numerical result (22b) provide any clues for the underlying puzzle of DLA? We suspect the answer to this question is "yes", and we have proposed a "void-channel" model of DLA [8,9] in order to explain the result (22b). The void-channel model states that each fjord is characterized by a hierarchy of voids separated from each other by narrow "channels" or "gateways." The key feature of the model are:

(1) The voids must be *self-similar*, i.e., their characteristic linear dimension must increase with the same exponent. Thus

$$L_{\rm void} \sim M^{1/d_f}. \tag{23}$$

To see this, we assume the contrary: if (23) does not hold, then DLA will not be a fractal!

(2) The voids are separated by channels or gateways: a random walker can pass form one void to the next only by passing through a gateway. If the diameter of a gateway also scales as M^{1/d_f}, then we would expect that $p_{\rm min}$ is given by the Harris *Ansatz*. Since the numerics do not support the Harris *Ansatz*, we conclude that [8,9]

$$L_{\rm channel} \sim M^{\gamma}. \qquad [\gamma < 1/d_f] \tag{24}$$

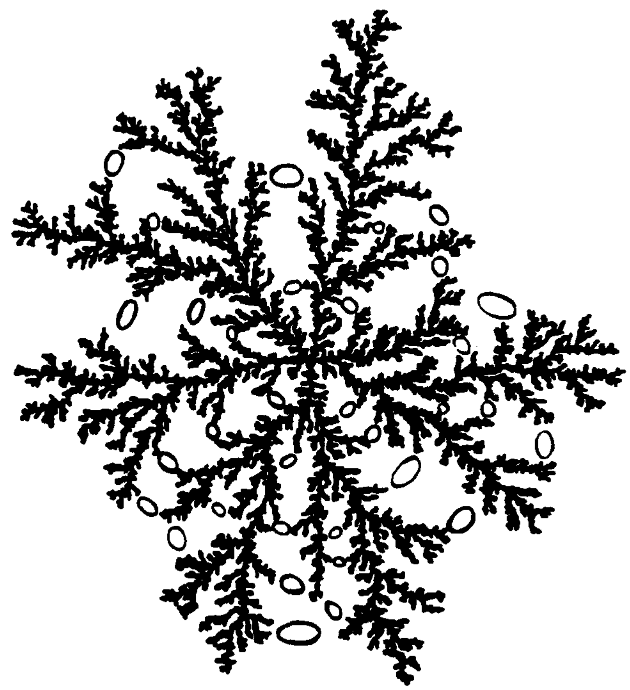

Fig. 11. Off-lattice DLA cluster of 10^5 sites indicating some of the channels that serve to delineate voids. Courtesy of P. Meakin.

What is the evidence supporting the void-channel model of DLA growth dynamics?

(1) First, we note that if channels "dominate", then the BA *Ansatz* would have to be satisfied. The numerics rule this out.

(2) Second, we note that if self-similar voids dominate, then the Harris *Ansatz* would have to be satisfied. Again, the numerics rule this out.

(3) Photos of large DLA clusters reveal the presence of such voids and channels [Fig. 11]. Moreover, when the DLA mass is doubled, we find that outer branches "grow together" to form new channels (enclosing larger and larger voids).

(4) The void-channel model can be *solved* [8,9] under the approximation that the voids are strictly self-similar and the gates obey (24). The solution demonstrates that $\log p_{\min} \propto (\log M)^2$.

(5) The void-channel model is consistent with a recent calculation [46] suggesting that DLA structures can be partitioned into two zones:

  (a) An inner *finished zone*, typically with $r \leq R_g$ (where R_g is the radius of gyration), for which the growth is essentially "finished" in the sense that it is overwhelmingly improbable that future growth will take place.

  (b) An outer *unfinished zone* (typically $r \geq R_g$) in which the growth is unfinished.

Thus future growth will almost certainly take place in the region $r > R_g$. Now $2R_g \approx \frac{1}{2}L$, where L is the spanning diameter. Hence only about 1/4 the total "projected area" of DLA is finished, the rest of the DLA being ***unfinished***. We suggest that the finished region will be created from the unfinished region by tips in the unfinished region growing into juxtaposition (thereby forming voids).

It remains to demonstrate numerically that *real* DLA is characterized by a hierarchy of self-similar voids separated by channels whose width scales with a power of L smaller than L^{1/d_f}. Such a numerical calculation is underway, using a hierarchy of *off-lattice* DLA clusters studied for a sequence of masses $M_o = 2^5, 2^6, \ldots, 2^{20}$. Although final numerical results are not available at the time of this talk, we do have visual evidence that outer tips widely separated for mass M_0 later grow together as the cluster size doubles and quadruples. Indeed, such behavior is expected since the growth of DLA is fixed by the growth probabilities, which are of course largest on the tips.

Two tips will grow closer and closer until their growth probabilities become so small that no further narrowing will occur. This observed phenomenon can be perhaps better understood if one notes that the growth probabilities $\{p_i\}$ of a given DLA cluster are identical to normalized values of the electric field $\{E_i\}$ on the surface of a charged conductor whose shape is identical to the given DLA cluster. Thus as two arms of the DLA "conductor" grow closer to each other, the electric field at their surface must become smaller (since $E_i \propto \nabla\phi_i$, where $\phi \equiv$ constant on the surface of the conductor). That E_i is smaller for two arms that are close together can be graphically demonstrated by stretching a drumhead* with a pair of open scissors.

(1) If the opening is big, the tips of the scissors are well-separated and the field on the surface is big (we see that the gradient of the altitude of the drumhead is large between the tips of the scissors).
(2) On the other hand, if the scissor tips are close together, the field is small (we see that the gradient of the altitude of the drumhead is small between the scissor tips).

5. Summary

In summary, we have (1) one "firm" numerical result, $\log p_{\min} \sim (\log M)^2$, given by Eq. (22b). We have also (2) an analytic argument that this behavior follows from a void-channel model of DLA structure in which there exist self-similar voids separated by channels whose width does not scale. We have (3) a plausibility argument that the tips of DLA grow together until they are separated by a distance which is typically a few pixels, and we have (4) visual evidence supporting this picture. We are presently working on obtaining firm numerical evidence to test the void-channel model of DLA growth dynamics.

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* A convenient drumhead is obtained by stretching panty hose across a circular sewing hoop (R. Selinger, private communication)

• Appendix: The Domb-Hunter Scaling Hypothesis

DLA is one model system for which the Domb-Hunter constant-gap hypothesis does not hold. The purpose of this Appendix is to describe the scaling hypothesis in this context, and to enquire if we can develop a deeper understanding of it from the behavior of DLA. We begin by considering a few simple examples:

• Example 1: Unbiased Random Walk

Consider, e.g., the Bernoulli probability distribution $\Pi(x,t)$, which gives the probability that a one-dimensional unbiased random walk is at position x at time t given that it was at $x = 0$ at time $t = 0$,

$$\Pi(x,t) = \binom{t}{\frac{x+t}{2}} \left(\frac{1}{2}\right)^t . \tag{A.1}$$

This distribution is characterized by its moments,

$$Z_\beta \equiv \langle x^\beta \rangle \equiv \sum_{x=-t}^{t} x^\beta \Pi(x,t). \tag{A.2}$$

Hence [47]

$$\langle x^o \rangle = 1 = t^o \tag{A.3a}$$

$$\langle x^2 \rangle = t \tag{A.3b}$$

$$\langle x^4 \rangle = 3t^2 - 2t = 3t^2 \left[1 - \frac{2}{3}\frac{1}{t}\right] \sim t^2 \tag{A.3c}$$

$$\langle x^6 \rangle = 15t^3 \left[1 - \frac{2}{t} + \frac{16}{15t^2}\right] \sim t^3 \tag{A.3d}$$

These moments have the property that if we write

$$Z_\beta \sim t^{-F(\beta)}, \tag{A.4}$$

then $F(\beta) = \beta/2$. The "Domb-Hunter gap"

$$\Delta(\beta) \equiv F(\beta+1) - F(\beta) \tag{A.5a}$$

is independent of β,

$$\Delta = 1/2 \qquad \text{[unbiased random walk]} \tag{A.5b}$$

. More generally, for a random walk with fractal dimension d_w, one can show that (A.5b) becomes

$$\Delta = 1/d_w \qquad \text{[fractal substrate]} \tag{A.5c}$$

•

• Example 2: Percolation

A second example is percolation. For an infinite system, Stauffer's scaling hypothesis [42] is well verified:

(a) Right at the the percolation threshold $p = p_c$, the system is self-similar on all length scales so the number of s-site clusters per lattice site decreases with s as a power law,

$$n_s \sim s^{-\tau}. \tag{A.6}$$

A remarkable fact is that the critical exponent τ controlling this decrease in the distribution of cluster sizes is directly connected to the fractal dimension d_f of the incipient infinite cluster,

$$\tau = 1 + d/d_f. \tag{A.7}$$

(b) Away from p_c, the system remains self-similar for length scales less than the pair connectedness length ξ. Hence the power law relation (A.6) must hold for all s smaller than a characteristic cluster size $s*$, where $s*$ is connected to ξ through

$$s* \sim \xi^{d_f} \sim |p - p_c|^{-\nu d_f}. \tag{A.8}$$

For values of s above $s*$, the system ceases to be self-similar and hence (A.6) must break down: the long-tail behavior of the power law (associated with "scale-free" behavior of a self-similar system) must cross over to a function with an inherent scale. That scale is, of necessity, set by $s*$ itself, so that when $p \neq p_c$ (A.6) must be replaced by

$$n_s(p) \sim n_s(p_c) f(s/s*). \tag{A.9}$$

The function $f(x)$ is sometimes called a *cut-off function* because it "cuts off" the power law of (A.6) above values of x where the system ceases to be self-similar. For the limit of infinite dimension d, the Cayley tree solution is believed to be exact and we know the explicit form of $f(x)$,

$$f(x) \sim \text{const} \qquad [x \ll 1], \tag{A.10a}$$

and

$$f(x) \sim \exp(-x^2) \qquad [x \gg 1]. \tag{A.10b}$$

For a system of edge L *at* the percolation threshold $p = p_c$, the basic quantity $n_s(p)$ is replaced by $n_s(L) \equiv N_s(L)/L^d$, where $N_s(L)$ is the number of clusters of s sites. On length scales much less than L, the system must be self-similar. Hence the analog of (A.9) is

$$n_s(L) \sim n_s(L = \infty) g(s/s*) \tag{A.11}$$

where $n_s(L = \infty) \sim s^{-\tau}$ and now $s* \sim L^{d_f}$. We say that the cluster size distribution is scale-free for cluster sizes smaller than $s*$ since on small length scales the system cannot "know" that it is finite.

The analog of (A.2) is

$$Z_\beta \equiv \langle s^\beta \rangle \equiv \sum_{s=1}^{\infty} s^\beta \Pi_s(L) = \sum_{s=1}^{\infty} s^{\beta+1} n_s(L). \tag{A.12}$$

since the probability that an arbitrarily chosen site belongs to an s-site cluster is given by $\Pi_s(L) \equiv sn_s(L)$. The scaling properties of the moments of this distribution then follow by dimensional analysis, since n_s has the dimensions of $s^{-\tau} \sim L^{-\tau d_f}$ and $\tau = 1 + d/d_f$. Thus

$$Z_{-1} \equiv \langle s^{-1} \rangle \equiv \sum_s s^{-1} \Pi_s(L) = \sum_s n_s(L) \sim L^{-d} \tag{A.13a}$$

$$Z_o \equiv \langle s^o \rangle \equiv \sum_s \Pi_s(L) = \sum_s sn_s(L) \sim L^{-(d-d_f)} \tag{A.13b}$$

$$Z_1 \equiv \langle s^1 \rangle \equiv \sum_s s^1 \Pi_s(L) = \sum_s s^2 n_s(L) \sim L^{-(d-2d_f)} \tag{A.13c}$$

Since $F(\beta)$ is defined by (13), we have

$$F(-1) = d \tag{A.14a}$$

$$F(0) = (d - d_f) \tag{A.14b}$$

$$F(1) = (d - 2d_f) \tag{A.14c}$$

We see from (A.14) that the family of "Domb-Hunter gap exponents" $\Delta(\beta) \equiv F(\beta+1) - F(\beta)$ collapses to a single value,

$$\Delta = -d_f. \qquad \text{[percolation]} \tag{A.14d}$$

Comparing (A.14d) with (A.5b) we see that percolation has the same simplifying feature which we found for the case of the simple random walk, namely the Domb-Hunter gap exponents are constant. Hence one needs to know only one exponent and the value of the gap exponent to determine *all* the exponents of the system.

That the Domb-Hunter formulation of scaling leads to the scaling equalities among the critical exponents for thermodynamics is demonstrated quite clearly in the original Domb-Hunger paper[1]. Here we demonstrate that fact for percolation by deriving the Rushbrooke equality

$$\alpha + 2\beta + \gamma = 2 \tag{A.15}$$

relating the critical exponents $2 - \alpha$ for the total number of clusters [Z_{-1} of Eq. (A.13a)], β (not to be confused with the β appearing in the moment expressions) for the fraction of sites belonging to a finite cluster [Z_o of (A.13b)], and γ for the average size of a finite cluster [Z_1 of (A.13c)]. From the definition (A.12) it follows that

$$F(-1) = \frac{2-\alpha}{\nu} \tag{A.16a}$$
$$F(0) = \beta/\nu \tag{A.16b}$$
$$F(1) = -\gamma/\nu \tag{A.16c}$$

so that

$$\Delta(-1) = \beta/\nu - \frac{2-\alpha}{\nu}, \tag{A.17a}$$
$$\Delta(0) = -\gamma/\nu - \beta/\nu. \tag{A.17b}$$

From (A.17) we see that the Domb-Hunter hypothesis $\Delta(0) = \Delta(1)$ leads immediately to the Rushbrooke exponent equality (A.15).

• Example 3: Correlated Spatial Disorder and Random Multiplicative Process

DLA is a disorderly growth model, just as is invasion percolation. Unlike percolation, for DLA there is a strong spatial correlation in the position of the particles. For DLA, an observation window centered on a "tip site" sees a quite different structure of growth probabilities than an observation window centered on a "fjord site." We say that the disorder in DLA is *spatially correlated*.

Is there a relation between the correlated spatial disorder of DLA and the breakdown of Domb-Hunter scaling? To try to answer this question, we now consider an extremely simple model of correlated spatial disorder, which, although extremely simple, differs fundamentally from other models of spatial disorder which generally take the spatial order to be random (e.g., by introducing random bias fields which alternate from point to point in the system). To study physical properties such as transport, most previous work has been based on variations of the classic percolation model in which the disordered material is treated as an *uncorrelated* network of random bonds (e.g, resistors) that are either open or blocked (finite or infinite resistivity). Thus the spatial disorder is assumed to be completely uncorrelated. However in many real disordered materials, such as polymers, porous materials, and amorphous systems, the spatial disorder is correlated. For example, if we model the permeability of a porous rock by an array of resistors whose resistances are chosen randomly, then it is possible to find huge resistances neighboring tiny resistances. Such configurations cannot occur in nature since the permeability of a "crack," while random, cannot fluctuate arbitrarily. The spatial disorder is *correlated*.

Reference [48] introduces a topologically one-dimensional model that encompasses the essential physics of *correlated* spatial disorder but is simple enough to be treated analytically. Consider a set of N resistors in series, where the resistance R_j of resistor j changes in a *correlated* fashion,

$$R_{j+1} \equiv (1+\epsilon)^{r_j} R_j. \tag{A.18}$$

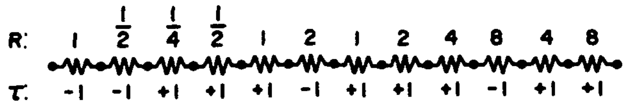

Fig. 12. A realization of the one-dimensional model for correlated spatial disorder (from Ref. [48])

Here $\epsilon > 0$ is arbitrary, and τ_j is chosen randomly to be $+1$ or -1 (see Fig. 12). Because neighboring resistors may only differ by a factor of $(1+\epsilon)$, this model insures a smooth spatial variation of the resistance.

If $\epsilon = 2$, then we have a simple one-dimensional random resistor network in which each resistance in the chain is either *twice* the preceding resistance or *half* the preceding resistance [48]. If we begin the chain with a unit resistor $R_o = 1$, then we choose the next resistor R_1 to be $R_1 = 2$ with probability $1/2$ or $R_1 = 1/2$ with probability $1/2$. There are clearly 2^N configurations of a chain of $N+1$ resistors.

One question of interest is the distribution for R_N, the resistance of resistor N. Clearly

$$(R_N)_{\max} = 2^N \tag{A.19a}$$

and

$$(R_N)_{\min} = (1/2)^N \tag{A.19b}$$

Moreover the distribution $\mathcal{D}(R_N)$ is quite *asymmetric* (like Fig. 5), since it has a maximum at the most probable value of R_N—which is unity—and a long tail extending to $(R_N)_{max} = 2^N$. Corresponding to this long tail is a set of moments that do *not* satisfy Domb-Hunter constant-gap scaling, since there are configurations with large values of R_N which dominate the moment sum. To make this explicit, note that for the first moment we have on summing on all 2^N configurations,

$$Z_1 \equiv \sum_{2^N c} [R_N(c)]\ P(c) = (5/4)^N, \tag{A.20a}$$

while for the second moment

$$Z_2 \equiv \sum_{2^N c} [R_N(c)]^2\ P(c) = (17/8)^N. \tag{A.20b}$$

Here $P(c)$ is the probability of each of the 2^N configurations c, and $R_N(c)$ is the value of resistor N in configuration c. Since $(17/8)^N$ is much larger than $(5/4)^{2N} = (25/16)^N$, we see that $Z_2 >> (Z_1)^2$.

Are multifractal phenomena associated with systems where the underlying physics is governed by a random *multiplicative* process? Certainly there are no multifractal phenomena associated with simple random *additive* processes (such as the sum of 8 numbers, each number being chosen to be either a -1 or

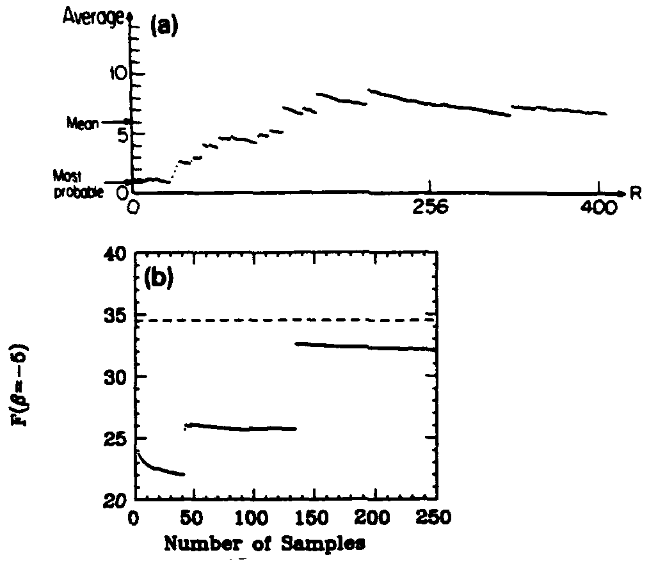

Fig. 13. (a) The results of a computer simulation of a random multiplicative process in which a string of 8 numbers is multiplied together, and each number is chosen with equal probability to be either 2 or 1/2. The limiting or asymptotic value of the product is $(5/4)^8 =$ 3.47. However the simulations do not give this value unless the number of realizations $\mathcal{R}$ is approximately the same as the total number of configurations of this product, $2^8 = 256$. This simulation was provided by R. Selinger. (b) Estimation of $F(\beta = -5, L = 4)$ obtained by random sampling for DLA grown in a 4x4 cell. The dashed lines are the exact value for $L = 4$. The running average shows 'jumps', and become close to the exact value after the number of samples is the same order of magnitude as the total number of configurations (259 in this case). This discontinuous jumps arise from the samples which give the dominant contribution to $F(\beta, L)$, but are very rare. From Ref. [2].

a +1—which has a geometrical interpretation as an 8-step random walk on a one-dimensional lattice).

To answer this question, consider a simple random *multiplicative* processes in which we form the product of 8 numbers, each number randomly chosen to be either a 1/2 or a 2 [49]. The results of simulations of such a process are shown in Fig. 13. The y-axis is the value of the product after $\mathcal{R}$ realizations, and the x-axis is the number of realizations $\mathcal{R}$. In total there are $2^8 = 256$ possible configurations of such random products. Normally, random sampling procedures give approximately correct answers when only a small fraction of the possible 256 configurations has been realized. Here, however, one sees from Fig. 13 that the correct asymptotic value of the product is attained only after approximately 256 realizations [24]. Monte Carlo sampling of only a small fraction of the 256 configurations is doomed to failure because of the 256 configurations, a rare few (consisting of, say, all 2's or seven 2's and a single 1) bias the average significantly and give rise to the upward steps in the running average shown in Fig. 13.

A simple random multiplicative process that gives rise to multifractal phenomena is found in the simple hierarchical model of the percolation backbone [50]. If the potential drop across the singly connected links is V_1 and that across the multiply-connected links is V_2, then we see that when this structure is iterated the potential drops across each of the bonds are products of the potential drops of the original structure. The reader can readily demonstrate that for this hierarchical structure $Z(\beta) = (V_1^\beta + V_2^\beta)^N$, where N is the number of iterations carried out [50]. It turns out that $Z(\beta)$ obeys a power law relation of the form of (13), with an infinite hierarchy of exponents given by $F(\beta) = 1 + \log\left[V_1^\beta + V_2^\beta\right] / \log 2$. In order to obtain this result, one must use the relation $N_{\rm red} \sim L^{3/4}$, where $N_{\rm red}$ is the number of singly-connected "red" bonds [42].

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[47] D. Stauffer and H. E. Stanley, *From Newton to Mandelbrot: A Primer in Modern Theoretical Physics* (Springer Verlag, Heidelberg, 1990).
[48] S. Havlin, R. Selinger, M. Schwartz, H.E. Stanley and A. Bunde, Phys. Rev. Letters **61**(1988) 1438.
[49] S. Redner, Am. J. Phys. **58** (1990) 267.
[50] L. de Arcangelis, S. Redner and A. Coniglio, Phys. Rev. B **34** (1986), 4656; A. Coniglio, *Physica A* **140** (1986) 51.

PATENTS FILED

NONE

PATENTS GRANTED

NONE

HONORS, INVITED TALKS AND PUBLICATIONS IN GRANT PERIOD

AWARDS & HONORS RECEIVED DURING THE GRANT PERIOD

(1) Thirtieth Saha Memorial Lecture, 1991 (delivered 5 Jan. 1992). [half the previous lecturers are Nobelists]

(2) December 1990: Science Citation Index recognition for Top 100 most-cited articles of 1988 [for A. Aharony, R. J. Birgeneau, A. Coniglio, M. A. Kastner and H. E. Stanley, "Magnetic Phase Diagram and Magnetic Pairing in Doped La_2CuO_4," Phys. Rev. Lett. **60**, 1330-1333 (1988).].

(3) Member, International Scientific Organizing Committee, *IUPAP International Conference on Statistical Mechanics (STATPHYS-17)*, Rio de Janeiro, August 1989.

(4) Co-Director, NATO Advanced Study Institute, *Propagation of Correlations in Constrained Systems*, Cargèse, France, 2-14 July 1990.

(5) Co-Director, International workshop on *Random Materials & Processes*, University of the West Indies, 18-22 December 1990.

(6) Member, International Scientific Organizing Committee, *IUPAP International Conference on Statistical Mechanics (STATPHYS-18)*, Berlin, August 1992.

INVITED TALKS DURING THE GRANT PERIOD

(7) Invited talk, *International Conference on Conductance of Disordered Materials*, Bar-Ilan University, 3-7 January 1987.

(8) Invited talk, *International Conference on the Physics of Chaos and Systems Far from Equilibrium*, Monterey, 11-14 January 1987.

(9) Invited talk, *International Conference on Physics and Engineering of Crystal Growth*, Pittsburgh, PA, 21-24 February 1987.

(10) Invited talk, *Bioengineering Conference*, University of Pennsylvania, 10-15 March 1987.

(11) Invited talk, American Physical Society, *Course on Polymer Physics*, NYC, 15-20 March 1987.

(12) Invited talk, *NATO Advanced Study Institute: Dynamic Aspects of Conductance in Disordered Materials*, Geilo, Norway (29 March – 10 April 1987).

(13) Invited talk, *International Conference on Physics of Porous Materials*, Trieste, Italy, 15-27 August 1987.

(14) Invited talk, *International Conference on the Microphysics of Enzyme Catalysis*, Lucca, Italy, 5-9 October 1987.

(15) Invited speaker, American Physical Society Semi-Annual Symposium, *Physics of Fluids*, Buffalo, NY, 16-17 October 1987.

(16) Invited speaker, *International Workshop on Polymer Dynamics*, Los Alamos National Laboratory, 2-5 November 1987.

(17) Invited speaker, International Conference on *Universalities in Condensed Matter Physics*, Les Houches, France, 14-25 March 1988.

(18) Invited speaker, American Physical Society, New Orleans, LA, 24-28 March 1988.

(19) Invited speaker, American Physical Society, Baltimore, MD 18-21 April 1988.

(20) Invited talk, *International Workshop on Nonequilibrium Patter Formation in Growth,* Johns Hopkins Univ, Baltimore, MD, 15-17 June 1988.

(21) Plenary Lecture, International Conference on *Nonlinear Variability in Geophysics: Fundamentals and Applications*, Paris, 27 June-1 July 1988.

(22) Invited speaker, *NATO Advanced Study Institute on Random Fluctuations and Pattern Growth*, Cargèse, France, 18-30 July 1988.

(23) Invited Participant, IBM Europe Workshop of High-T_c-Temperature Superconductivity (K. A. Müller and G. Bednosz, Directors) 8-12 August 1988.

(24) Invited speaker, *International Seminar on Cooperative Dynamics in Complex Physical Systems*, Kyoto, Japan, 24-27 August 1988.

(25) Invited lecture, International Conference ETOPIM *(Electrical Transport and Optical Properties of Inhomogeneous Media)*, Paris, France 28 August – 2 September 1988.

(26) Invited Talk, *International Conference on Fractals in Physics* Erice, Sicily. 10-14 October 1988.

(27) Invited Talk, *International Workshop on Statistical Physics in Geology*, Asilomar, CA, 11-15 February 1989.

(28) Invited Talk, *American Physical Society*, St. Louis, MO 20-24 March 1989.

(29) Invited Talk, *American Chemical Society*, Dallas, TX 10-14 April 1989.

(30) Invited Talk, IUPAP International Conference on Statistical Mechanics (STATPHYS-17), Rio de Janeiro, 31 July – 4 August 1989.

(31) Invited Talk, Ninth International Conference on Crystal Growth, Sendai, Japan 20-25 August 1989.

(32) Invited Talk, *Fractals in Physics: In Honor of the 65th Birthday of Professor Benoit Mandelbrot*, Nice, France, 1-4 October 1989.

(33) Invited Talk, *International Conference on Frontiers in Condensed Matter Physics*, Bar Ilan University, Israel, 8-11 January 1990.

(34) Invited Talk, *Physics, Chemistry and Materials Science of Clusters*, Lake Arrowhead, California, 21-23 January 1990.

(35) Invited Talk, *FASEB Meeting*, Washington, DC, 1 April 1990.

(36) Keynote Address, *MECO-17 (17th Conference of the Middle-European Cooperation in Statistical Physics)*, Balatonfüred, Hungary, 22-25 April 1990.

(37) Invited Talk, *Venture Research Conference*, London, UK, 25-27 June 1990.

(38) Invited Talk, *NATO Advanced Study Institute on Propagation of Correlations in Constrained Systems*, Cargèse, France, 2-14 July 1990.

(39) Invited Talk, *Complexity in Physics: Entering the 21st Century*, Stockholm, Sweden, 3-8 September 1990.

(40) Opening Talk, *Random Materials & Processes*, University of West Indies, 18-23 December 1990.

BOOKS PUBLISHED DURING THE GRANT PERIOD

1. H. E. Stanley and N. Ostrowsky, Eds. *Random Fluctuations and Pattern Growth: Experiments & Theory* (Proceedings 1988 Cargèse NATO ASI Series E: Applied Sciences, Vol 157). Kluwer Academic Publishers, Dordrecht, 1988.
2. H. E. Stanley and N. Ostrowsky, eds., *Correlations and Connectivity: Geometric Aspects of Physics, Chemistry and Biology* (Kluwer, Dordrecht, 1990). [PROC. 1990 NATO ADVANCED STUDY INSTITUTE.]
3. D. Stauffer and H. E. Stanley, *From Newton to Mandelbrot: A Primer in Theoretical Physics* (Springer-Verlag, Heidelberg & New York, 1990).

BOOK CHAPTERS, PUBLICATIONS, AND CONF. PROC. DURING THE GRANT PERIOD

4. A. Bunde, S. Miyazima, and H. E. Stanley, "A growth model with a finite lifetime of growth sites: From the Eden model to the kinetic growth walk" J. Stat. Phys. **47**, 1-16 (1987).
5. A. Coniglio, N. Jan, I. Majid and H. E. Stanley, "New model embodying the physical mechanism of the coil-globule transition at the theta point of a linear polymer" Phys. Rev. B **35**, 3617-3620 (1987).
6. J. Nittmann, H. E. Stanley, E. Touboul, and G. Daccord, "Experimental Evidence for Multifractality," Phys. Rev. Lett. **58**, 619 (1987).
7. J. Nittmann and H. E. Stanley, "Non-Deterministic Approach to Anisotropic Growth Patterns with Continuously-Tunable Morphology: The Fractal Properties of Some Real Snowflakes," J. Phys. A **20**, L1185 (1987).
8. H. E. Stanley, D. Stauffer, J. Kertész and H. J. Herrmann, "Dynamics of Spreading Phenomena in Cooperative Models," Phys. Rev. Lett. 2326-2328 (1987).

9. D. Marković, S. Milošević, and H. E. Stanley, "Self-Avoiding Walks on Random Networks of Resistors and Diodes" Physica A **144**, 1-16 (1987).

10. S. Miyazima and H. E. Stanley, "Intersection of Two Fractal Objects: Useful Method of Estimating the Fractal Dimension," Phys. Rev. B (Rapid Comm.) **35**, 8898 (1987).

11. H. E. Stanley and S. Havlin, "Generalization of the Sinai Anomalous Diffusion Law," J. Phys. A **20**, L615-L618 (1987).

12. P. Devillard and H. E. Stanley, "First-order branching in diffusion-limited aggregation," *Phys. Rev. A* **36**, 5359 (1987).

13. Z. V. Djordjevic and H. E. Stanley, "Scaling properties of the perimeter distribution for lattice animals, percolation and compact clusters," J. Phys. A **20**, L587-L594 (1987).

14. J. Nittmann and H. E. Stanley, "Role of Fluctuations in Viscous Fingering and Dendritic Crystal Growth: A Noise-Driven Model with Non-Periodic Sidebranching and No Threshold for Onset" *J. Phys. A* **20**, L981-L986 (1987).

15. A. Bunde, S. Miyazima, and H. E. Stanley, "From the Eden model to the kinetic growth walk: A generalized growth model with a finite lifetime of growth sites" J. Stat. Phys. **47**, 1-16 (1987).

16. P. Poole, A. Coniglio, N. Jan, and H. E. Stanley, "Universality Classes for the θ and θ' Points" Phys. Rev. Lett. **60**, 1203 (1988).

17. P. Alstrøm, D. Stassinopoulos and H. E. Stanley, "'Thermodynamical Formalism' for an Infinite Hierarchy of Fractal Resistor Networks" Physica A **153** 20-46 (1988).

18. T. Stošić, B Stošić, S. Milošević and H. E. Stanley, "Crossover from Fractal Lattice to Euclidean Lattice for the Residual Entropy of an Ising Antiferromagnet in Maximum Critical Field H_c," Phys. Rev. A **37** 1747-1753 (1988).

19. A. Aharony, R. J. Birgeneau, A. Coniglio, M. A. Kastner and H. E. Stanley, "Magnetic Phase Diagram and Magnetic Pairing in Doped La_2CuO_4," Phys. Rev. Lett. **60**, 1330 (1988).

20. A. Aharony, R. J. Birgeneau, A. Coniglio, M. A. Kastner and H. E. Stanley, "Magnetic Phases and Possible Magnetic Pairing in Doped Lanthanum Cuprate," Physica C **153-155**, 1211 (1988).

21. A. Bunde, S. Havlin, H. E. Roman, G. Schildt and H. E. Stanley, "On the Field Dependence of Random Walks in the Presence of Random Fields," J. Stat. Phys. 50, 127 (1988).

22. E. Koscielny-Bunde, A. Bunde, S. Havlin, and H. E. Stanley, "Diffusion in the Presence of Random Fields and Transition Rates: Effect of the Hard Core Interaction," Phys. Rev. A **37**, 1821-1823 (1988).

23. S. Milošević, D. Stassinopoulos, and H. E. Stanley, "Asymptotic Form of the Spectral Dimension at the Fractal to Lattice Crossover" J. Phys. A **21** 1477-1482 (1988).

24. S. Havlin, R. Selinger, M. Schwartz, H.E. Stanley and A. Bunde "Random multiplicative processes and transport in structures with correlated spatial disorder", Phys. Rev. Letters **61**, 1438 (1988).

25. S. Miyazima, Y. Hasegawa, A. Bunde and H. E. Stanley, "Generalized Diffusion-Limited Aggregation Where Aggregate Sites have a Finite Radical Time," J. Phys. Soc. Japan **57**, 3376-3380 (1988).

26. T. Stošić, B. Stošić, S. Milošević and H. E. Stanley, "Crossover from Fractal Lattice to Euclidean Lattice for the Residual Entropy of an Ising Antiferromagnet in Maximum Critical Field H_c," Phys. Rev. A **37** 1747-1753 (1988).

27. J. Lee and H. E. Stanley "Phase Transition in the Multifractal Spectrum of Diffusion-Limited Aggregation," Phys. Rev. Lett. **61**, 2945-2948 (1988).

28. P. Devillard and H. E. Stanley, "Roughening for Diffusion Limited Aggregation with Walkers Having a Finite Lifetime" *Phys. Rev. A* **38**, 6451 (1988).

29. H. J. Herrmann and H. E. Stanley, "The fractal dimension of the minimum path in two- and three-dimensional percolation" J. Phys. A **21** L829-L833 (1988).

30. C. Amitrano, P. Meakin, and H. E. Stanley, "Fractal Dimension of the Accessible Perimeter of Diffusion-Limited-Aggregation" Phys. Rev. A **40**, 1713-1716 (1989).

31. E. Arian, P. Alstrøm, A. Aharony, and H. E. Stanley, "Crossover Scaling from Multifractal Theory: Dielectric Breakdown with Cutoffs," Phys. Rev. Lett. **63**, 2005 (1989).

32. A. Coniglio and H. E. Stanley, "Dilute Annealed Magnetism and High-Temperature Superconductivity," Physica C **161**, 88-90 (1989).

33. P. Devillard and H. E. Stanley, "Scaling Properties of Eden Clusters in Three and Four Dimensions," Physica A **160**, 298-309 (1989).

34. A. D. Fowler, H. E. Stanley and G. Daccord, "Disequilibrium Silicate Mineral Textures: Fractal and Non-Fractal Features," Nature **341**, 134-138 (1989).

35. S. Havlin, M. Schwartz, R. Blumberg Selinger, A. Bunde, and H. E. Stanley, "Universality Classes for Diffusion in the Presence of Correlated Spatial Disorder" Phys. Rev. A **40**, 1717-1719 (1989).

36. G. Huber, P. Alstrøm, and H. E. Stanley, "Number of Scaling Factors in Incommensurate Systems," J. Phys. A **22**, L279-L285 (1989).

37. J. Lee and H. E. Stanley, "Phase Transition in Diffusion-Limited Aggregations: Lee and Stanley Reply," Phys. Rev. Lett. **63**, 1190 (1989).

38. J. Lee, P. Alstrøm, and H. E. Stanley "An Exact Enumeration Approach to Multifractal Structure for Diffusion Limited Aggregation," Phys. Rev. A **39**, 6545-6556 (1989).

39. J. Lee, P. Alstrøm, and H. E. Stanley, "Scaling of the Minimum Growth Probability for the 'Typical' DLA Configuration," Phys. Rev. Lett. **62**, 3013 (19 June 1989).

40. P. Poole, A. Coniglio, N. Jan, and H. E. Stanley, "What is the Shape of a Polymer Chain near the Theta Point?" *Phys. Rev. A* **39**, 495 (1989).

41. F. Sciortino, P. Poole, H.E.Stanley and S. Havlin, "Lifetime of the Hydrogen Bond Network and Gel-Like Anomalies in Supercooled Water", Phys. Rev. Letters **64**, 1686-1689 (1990).

42. R. B. Selinger, S. Havlin, F. Leyvraz, M. Schwartz, and H. E. Stanley, "Diffusion in the Presence of Quenched Random Bias Fields: A Two-Dimensional Generalization of the Sinai Model," Phys. Rev. A **40**, 6755-6758 (1989).

43. R. B. Selinger, J. Nittmann, and H. E. Stanley, "Inhomogeneous Diffusion-Limited Aggregation," Phys. Rev. A **40**, 2590-2601 (1989).

44. H. E. Stanley, "Learning Concepts of Fractals & Probability by 'Doing Science'" Physica D **38**, 330-340 (1989).

45. B. Stošić and H. E. Stanley, "Low Temperature Impurity Pairing in the Frustrated 2d Ising Model," Physica A **160**, 148-156 (1989).

46. R. J. Vasconcelos dos Santos, I. P. Fittipaldi, P. Alstrøm and H. E. Stanley, "Exact Results for Randomly Decorated Magnetic Frustrated Models of Planar CuO_2 Systems," Phys. Rev. B **40**, 4527-4531 (1989).

47. P. Alstrøm, D. Stassinopoulos and H. E. Stanley, "Images and Distributions Obtained from Affine Transformations," Phys. Rev. A **41**, 5290 (1990).

48. J. Lee, P. Alstrøm, and H. E. Stanley, "Is There a Phase Transition in the Multifractal Spectrum of DLA?" in *Fractals: Physical Origin and Properties*, L. Pietronero, ed. (Plenum Publishing Co., London, 1990). [INVITED TALK, ERICE.]

49. H. E. Stanley, "Role of Fluctuations in Fluid Mechanics and Dendritic Solidification," Physica A **163**, 334-358 (1990). [INVITED TALK, STATPHYS 17.]

50. H. E. Stanley, A. Bunde, S. Havlin, J. Lee, E. Roman, and S. Schwarzer, "Dynamic Mechanisms of Disorderly Growth: Recent Approaches to Understanding Diffusion Limited Aggregation," Physica A **168**, 23-48 (1990). [INVITED TALK, INTERNATIONAL CONFERENCE ON FRONTIERS IN CONDENSED MATTER PHYSICS.]

51. A. Coniglio, "Correlations in Thermal and Geometrical Systems," in *Correlations and Connectivity: Geometric Aspects of Physics, Chemistry and Biology* [NATO ASI SERIES VOL. 188], eds. H. E. Stanley and N. Ostrowsky (Kluwer, Dordrecht, 1990).

52. P. Alstrøm and F. Sciortino, "Dynamics of Bonded Networks with Two Energy Scales," Phys. Rev. Lett. **65**, 2885 (1990).

53. P. Alstrøm, P. Trunfio, and H. E. Stanley, "Spatio-Temporal Fluctuations in Growth Phenomena: Dynamical Phases and 1/f Noise," Phys. Rev. A Rapid Communications **41**, 3403 (1990).
54. F. Caserta, H. E. Stanley, W. Eldred, G. Daccord, R. Hausman, and J. Nittmann, "Physical Mechanisms Underlying Neurite Outgrowth: A Quantitative Analysis of Neuronal Shape," Phys. Rev. Lett. **64**, 95-98 (1990).
55. P. Devillard and H. E. Stanley, "Exact Enumeration Approach to the Directed Polymer Problem," Phys. Rev. A **41**, 2942-2951 (1990).
56. J. Lee, A. Coniglio, and H. E. Stanley, "Fractal-to-Nonfractal Crossover for Viscous Fingers," Phys. Rev. A Rapid Communications **41**, 4589-4592 (1990).
57. J. Lee, S. Havlin, H. E. Stanley, and J. E. Kiefer, "Hierarchical Model for the Multifractality of Diffusion Limited Aggregation," Phys. Rev. A **42**, 4832-4837 (1990)
58. T. Nagatani and H. E. Stanley, "Double Crossover Phenomena in Laplacian Growth: Effects of Sticking Probability and Finite Viscosity Ratio," Phys. Rev. A **41**, 3263-3269 (1990).
59. T. Nagatani and H. E. Stanley, "Phase Transition and Crossover in Diffusion-Limited Aggregation with Reaction Times," Phys. Rev. A **42**, 3512-3517 (1990).
60. T. Nagatani and H. E. Stanley, "Crossover and Thermodynamic Representation in the Extended η Model for Fractal Growth," Phys. Rev. A **42**, 4838-4844 (1990).
61. C.-K. Peng, S. Prakash, H. J. Herrmann, and H. E. Stanley, "Randomness versus Deterministic Chaos: Effect on Invasion Percolation Clusters," Phys. Rev. A **42**, 4537-4542 (1990).
62. S. Schwarzer, J. Lee, A. Bunde, S. Havlin, H. E. Roman, and H. E. Stanley, "Minimum Growth Probability of Diffusion-Limited Aggregates," Phys. Rev. Lett. **65**, 603-606 (1990).
63. F. Sciortino, A. Geiger, and H. E. Stanley, "Isochoric Differential Scattering Functions in Liquid Water: The Fifth Neighbor as a Network Defect," Phys. Rev. Lett. **65**, 3452 (1990).
64. F. Sciortino, P. Poole, H. E. Stanley and S. Havlin, "Lifetime of the Hydrogen Bond Network and Gel-Like Anomalies in Supercooled Water," Phys. Rev. Lett. **64**, 1686-1689 (1990).
65. R. L. B. Selinger and H. E. Stanley, "Percolation of Interacting Diffusing Particles," Phys. Rev. A **42**, 4845-4852 (1990).
66. H. E. Stanley, P. Poole, A. Coniglio, and N. Jan, "What is the Shape of a Polymer Chain near the Theta Point?" Mol. Cryst. & Liquid Crst. **18**, 91-92 (1990).
67. D. Stassinopoulos, G. Huber, and P. Alstrøm, "Measuring the Onset of Spatio-Temporal Intermittency," Phys. Rev. Lett. **64**, 3007-3010 (1990).
68. B. Stošić, S. Milošević, and H. E. Stanley, "Density of States Functions of the Fully Finite Ising Model Systems," Phys. Rev. B **41**, 11466 (1990).
69. P. Trunfio and P. Alstrøm "Exponentially Small Growth Probabilities in Diffusion-Limited Aggregation," Phys. Rev. B **41**, 896-898 (1990).